Introduction to Olympiad Inequalities

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Contents

Abstract

This booklet is designed to briefly summarize a math course offered during the Spring HSSP 2017 organized by Educational Studies Program at Massachusetts Institute of Technology. It is designed for students with no prior experience with Olympiad Inequalities who have fundamental knowledge of middle school algebra. The main goal is to introduce the students with this common Math Olympiad topic and present them with creative and elegant methods of solving problems of similar kind.

1 Warm up and Am-Gm inequality

1.1 Elementary inequalities

- if $a \geq b$ and $b \geq c$ then $a \geq c$ for any $a, b, c \in \mathbb{R}$
- if $a \geq b$ then $a + c \geq b + c$ for any $a, b, c \in \mathbb{R}$
- if $a \geq b$ and $x \geq y$ then $a + x \geq b + y$ for any $a, b, x, y \in \mathbb{R}$
- if $a \geq b$ and $x \geq y$ then $ax \geq by$ for any $a, b, x, y \in \mathbb{R}^+$

1.2 Aritmetic Mean $\geq$ Geometric Mean

Let $a_1, a_2 \ldots, a_n$ be non-negative real numbers. Then, the following inequality holds:

$$\frac{a_1 + a_2 + ... + a_n}{n} \geq \sqrt[n]{a_1 a_2 \cdots a_n} \tag{1}$$

Equality holds for $a_1 = a_2 = ... = a_n$

1.3 Examples

Example 1 *Prove the following inequality for every $a, b, c \in \mathbb{R}^+$:*

$$\frac{a^2}{b^2} + \frac{b^2}{c^2} + \frac{c^2}{a^2} \geq \frac{b}{a} + \frac{c}{b} + \frac{a}{c}$$

Solution:
Applying AM-GM inequality on $\frac{a^2}{b^2}$ and $\frac{b^2}{c^2}$ gives us:

$$\frac{a^2}{b^2} + \frac{b^2}{c^2} \geq 2\sqrt{\frac{a^2 b^2}{b^2 c^2}} = 2\frac{a}{c}$$

This is awesome, because $\frac{a}{c}$ is one of the terms on the RHS of the inequality. Completely analogously we observe:

$$\frac{b^2}{c^2} + \frac{c^2}{a^2} \geq 2\sqrt{\frac{b^2 c^2}{c^2 a^2}} = 2\frac{b}{a}$$

and

$$\frac{c^2}{a^2} + \frac{a^2}{b^2} \geq 2\sqrt{\frac{c^2a^2}{a^2b^2}} = 2\frac{c}{b}$$

If we sum up these inequalities and divide by 2 we have:

$$\frac{a^2}{b^2} + \frac{b^2}{c^2} + \frac{c^2}{a^2} \geq \frac{b}{a} + \frac{c}{b} + \frac{a}{c}$$

QED
Note that equality occurs when $a = b = c$

Example 2 *Prove the following inequality for every $a, b, c \in \mathbb{R}^+$:*

$$(a + \frac{1}{b})(b + \frac{1}{c})(c + \frac{1}{a}) \geq 8$$

When does equality occur?
Solution:
Using AM-GM inequality we have:

$$a + \frac{1}{b} \geq 2\sqrt{\frac{a}{b}}, \qquad b + \frac{1}{c} \geq 2\sqrt{\frac{b}{c}} and \qquad c + \frac{1}{a} \geq 2\sqrt{\frac{c}{a}}$$

Therefore,

$$(a + \frac{1}{b})(b + \frac{1}{c})(c + \frac{1}{a}) \geq 8\sqrt{\frac{a}{b}}\sqrt{\frac{b}{c}}\sqrt{\frac{c}{a}} = 8$$

and that is what we needed to prove. Equality occurs when $a = \frac{1}{b}$, $b = \frac{1}{c}$ and $c = \frac{1}{a}$ leading to $a = b = c = 1$

Example 3 *Find all real solutions of the following system:*

$$x + y = 2$$

$$xy - z^2 = 1$$

Solution:
Even though it might seem that this problem has nothing to do with inequalities, we will show the power of using inequalities to solve many kinds of problems. Let's start by noting that $2 = x + y \geq 2\sqrt{xy}$. By doing some simple algebra, it is not hard to conclude that $xy \leq 1$. On the other hand, using

the second equation and the fact that $xy \leq 1$ *we get:* $1 = xy - z^2 \leq 1 - z^2$. *By cancelling out the ones on both sides, we get that* $0 \leq -z^2$ *Since every number squared is greater or equal to 0, it follows from the last inequality that* z *must be 0. Therefore, we also get that equality holds for* $x = y = 1$. *This is one example where inequalities can be used to solve other types of problems.*

2 Common identities and other means

2.1 Identities

There are many identities that problem solvers use in order to prove inequalities. They allow us to transform the inequality to another, equivalent inequality which is easier to prove. Here are some of the most used identities:

- $a^2 + b^2 + c^2 = (a + b + c)^2 - 2(ab + bc + ca)$
- $(a + b)(b + c)(c + a) = (a + b + c)(ab + bc + ca) - abc$
- $(1 + a)(1 + b)(1 + c) = 1 + (a + b + c) + (ab + bc + ca) + abc$
- $(a+b)(b+c)+(b+c)(c+a)+(c+a)(a+b) = (a+b+c)^2+(ab+bc+ca)$
- $ab(a + b) + bc(b + c) + ca(c + a) = (a + b + c)(ab + bc + ca) - 3abc$

2.2 Other Means: Quadratic Mean and Harmonic Mean

Until now we have familiarized ourselves with Arithmetic and Geometric Means and the inequality that holds between them. Now we will introduce another two means and we will see how they fit into the so-called Means inequality.

The Quadratic Mean for n numbers is defined as:

$$Q = \sqrt{\frac{a_1^2 + a_2^2 + ... + a_n^2}{n}}$$

The Harmonic Mean for n numbers is defined as:

$$H = \frac{n}{\frac{1}{a_1} + \frac{1}{a_2} + ... + \frac{1}{a_n}}$$

If we denote Arithmetic Mean and Geometric Mean with A and G respectively, we can introduce the following important result:

Means Inequality:

$$Q \geq A \geq G \geq H$$

Equality holds for $a_1 = a_2 = ... = a_n$

2.3 Examples

Example 4 *Let* $a, b, c, d > 0$ *such that* $a + b + c + d = 1$. *Prove the inequality:*

$$\frac{1}{4a + 3b + c} + \frac{1}{3a + b + 4d} + \frac{1}{a + 4c + 3d} + \frac{1}{4b + 3c + d} \geq 2\,.$$

Solution: Let $A = \frac{1}{4a + 3b + c}$, $B = \frac{1}{3a + b + 4d}$, $C = \frac{1}{a + 4c + 3d}$ *and* $D = \frac{1}{4b + 3c + d}$ ..

Using Am-Hm for A,B,C,D we get: $\Longrightarrow \quad \frac{A + B + C + D}{4} \geq \frac{4}{\frac{1}{A} + \frac{1}{B} + \frac{1}{C} + \frac{1}{D}}$

$\Longleftrightarrow \quad A + B + C + D \geq \frac{16}{\frac{1}{A} + \frac{1}{B} + \frac{1}{C} + \frac{1}{D}}$

$LHS \geq \frac{16}{(4a + 3b + c) + (3a + b + 4d) + (a + 4c + 3d) + (4b + 3c + d)} = \frac{16}{8a + 8b + 8c + 8d} = \frac{16}{8} = 2$

Example 5 *Let* $a, b, c, > 0$ *such that* $abc = 1$. *Prove the inequality:*

$$\frac{a}{(a + 1)(b + 1)} + \frac{b}{(b + 1)(c + 1)} + \frac{c}{(c + 1)(a + 1)} \geq \frac{3}{4}$$

Solution:
One of the useful strategies here would be to get rid of the denominators and hope to get something better. Let's multiply everything with $4(a+1)(b+1)(c+1)$*. We get the following:*

$$4a(c+1)+4b(a+1)+4c(b+1) \geq 3(a+1)(b+1)(c+1)$$

which is equivalent to:

$$4(ab+bc+ca+a+b+c) \geq 3(1+a+b+c+ab+bc+ca+abc)$$

Note that here we used one of the above mentioned identities to simplify the expression. By cancelling the like-terms we obtain that we are left to prove:

$$ab+bc+ca+a+b+c \geq 3(1+abc)$$

. If we substitute $abc = 1$ *(which is given in the statement of the problem) we are left to prove that*

$$ab+bc+ca+a+b+c \geq 3(1+1) = 6$$

. Note that the last inequality is true from AM-GM inequality applied on the six terms a, b, c, ab, bc, ca *and using once more the fact that* $abc = 1$*.*

3 Geometric (triangle) inequalities

3.1 Notation

There are some common notations when it comes to triangles:
a, b, c - sides of the triangle
A - Area
s- half-perimeter, namely $s = \frac{a+b+c}{2}$
R - circumradius (radius of the circumcircle)
r - inradius (radius of the incircle)

3.2 The basic triangle inequality and a common substitution

One of the most important things to note about triangles is that the sum of any two sides is ALWAYS greater than the third side. That means, if a, b, c

are sides of a triangle, then all of the following inequalities hold:

$$a + b > c$$

$$b + c > a$$

$$c + a > b$$

Another thing that we will often encounter while solving triangle inequalities is the fact that the sides of any triangle can be represented in the following way: $c = x + y$, $b = x + z$, $a = y + z$ where x, y, z are positive real numbers. This is called *Ravi transformation* and it holds true based on the properties of the inscribed circle in a triangle, as illustrated in figure 1.

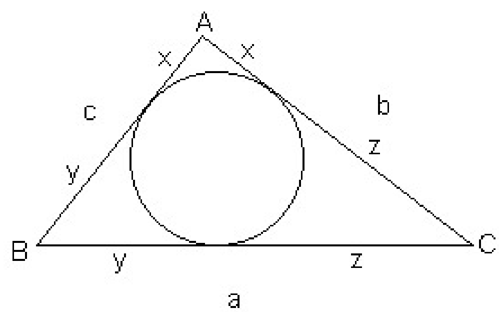

Figure 1: Triangle and its incircle.

Example 6 *Let a,b,c be sides of a triangle. Prove the following inequality:*

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} < 2$$

Solution:
Since a, b, c are sides of a triangle we have that $a + b > c$. If we add $a + b$ on both sides of the inequality we get $2(a + b) > a + b + c$. This is equivalent to $a + b > \frac{a+b+c}{2} = s$ i.e. $\frac{1}{a+b} < \frac{1}{s}$. By multiplying this with c we get $\frac{c}{a+b} < \frac{c}{s}$. This looks exactly like the third term on the left hand side of the inequality. Note that analogously we can get the other two terms. If we add these results we get that

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} < \frac{a+b+c}{s} = \frac{a+b+c}{\frac{a+b+c}{2}} = 2$$

which is exactly what we wanted to prove.

Example 7 *Let a,b,c be sides of a triangle. Prove the following inequality:*

$$(a + b - c)(b + c - a)(c + a - b) \leq abc$$

Solution:
Using the Ravi transformation we can make the following substitution: $a = x + y$, $b = y + z$, $c = z + x$. *Note that even though this is not identical as the substitution written above, the essence and meaning are completely the same. Note that from this we get:* $a + b - c = 2y$, $b + c - a = 2z$ *and* $c + a - b = 2x$. *Now the initial inequality looks like this:*

$$(x + y)(y + z)(z + x) \geq 8xyz$$

This proof of this last inequality is straightforward application of AM-GM inequality in each of the parentheses and multiplying them together, similar as in example 2.

4 Cauchy-Schwartz, Titu's lemma and Nesbitt's inequality

4.1 Cauchy-Schwartz inequality

Cauchy-Schwartz is one of the most common inequalities besides AM-GM. It is stated as follows:

Let $a_1, a_2 \ldots, a_n$ and $b_1, b_2 \ldots, b_n$ be real numbers. Then the following holds:

$$(a_1^2 + a_2^2 + \ldots + a_n^2)(b_1^2 + b_2^2 + \ldots + b_n^2) \geq (a_1b_1 + a_2b_2 + \ldots + a_nb_n)^2$$

with equality if and only if $\frac{a_1}{b_1} = \frac{a_2}{b_2} = \ldots = \frac{a_n}{b_n}$

While we will not give full proof of this theorem in this occasion, we highly encourage the reader to tryto prove the statement for $n = 2$.

4.2 Titu's lemma

Titu's lemma is a direct consequence of Cauchy-Schwartz inequality. It is also known as Cauchy-Schwartz inequality in Engel's form. It is an extremely useful shortcut for solving inequalities whose numerators are perfect squares. Let's take a look at it:

Let $a_1, a_2 \ldots, a_n$ and $b_1, b_2 \ldots, b_n$ be real numbers. Then the following holds:

$$\frac{a_1^2}{b_1} + \frac{a_2^2}{b_2} + \ldots + \frac{a_n^2}{b_n} \geq \frac{(a_1 + a_2 + \ldots + a_n)^2}{b_1 + b_2 + \ldots + b_n}$$

Equality holds for $\frac{a_1}{b_1} = \frac{a_2}{b_2} = \ldots = \frac{a_n}{b_n}$

The proof of this inequality is straightforward consequence of Cauchy-Schwartz inequality, which is easily observed if we multiply both sides of the inequality with $b_1 + b_2 + \ldots + b_n$.

4.3 Nesbitt's inequality

Nesbitt's inequality deserves a special place in the world of olympiad inequalities since it often appears as part of proving larger problems. Let's take a look at it:

Let a, b, c be positive real numbers. Then we have:

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq \frac{3}{2}$$

Equality is achieved for $a = b = c$

Proof:
There are many ways to prove this inequality, but here we will make use of Cauchy-Schwartz inequality.
It would be nice if we can make all numerators the same. To achieve this we will add the number 1 to each fraction separately, which is equivalent to adding 3 to both sides of the inequality. This yields to:

$$\frac{a}{b+c} + 1 + \frac{b}{c+a} + 1 + \frac{c}{a+b} + 1 \geq \frac{9}{2}$$

Note that this is equivalent to:

$$\frac{a+b+c}{b+c} + \frac{a+b+c}{c+a} + \frac{a+b+c}{a+b} \geq \frac{9}{2}$$

or

$$2(a+b+c)(\frac{1}{b+c}+\frac{1}{c+a}+\frac{1}{a+b}) \geq 9$$

which can also be written as

$$((b+c)+(c+a)+(a+b))(\frac{1}{b+c}+\frac{1}{c+a}+\frac{1}{a+b}) \geq 9$$

Note that if we treat $b+c$ as a_1, $\frac{1}{b+c}$ as b_1 in the definition above and analogously with the other ones, we can apply Cauchy-Schwartz inequality easily to get that the left hand side of the last inequality is greater than or equal to $(1+1+1)^2$ which is just 9 as we needed to prove. *Note:* We could have proven the last inequality very easily using AM-HM! Think about how would you do that.

4.4 Examples

Example 8 *Let a, b, c be real numbers. Prove the inequality:*

$$2a^2+3b^2+6c^2 \geq (a+b+c)^2$$

Solution:
While the numbers on the left hand side seem unrelated and intimidating, they have one property which immediately solves the inequality with the help of Cauchy-Schwartz inequality. Namely $\frac{1}{2}+\frac{1}{3}+\frac{1}{6}=1$. Now we have the following:

$$LHS = 1 \cdot (2a^2+3b^2+6c^2) = (\frac{1}{2}+\frac{1}{3}+\frac{1}{6})(2a^2+3b^2+6c^2) \geq (a+b+c)^2$$

and that is what we needed to prove.

Example 9 *Let x, y, z be positive real numbers. Prove the following inequality:*

$$\frac{2}{x+y}+\frac{2}{y+z}+\frac{2}{z+x} \geq \frac{9}{x+y+z}$$

Solution:
If we look closely at the denominators, we might be able to see that applying Titu's lemma should be helpful in this case. The fact that the nominators

are not perfect squares shouldn't discourage us, since we can make them to be perfect squares! We have the following:

$$\frac{(\sqrt{2})^2}{x+y} + \frac{(\sqrt{2})^2}{y+z} + \frac{(\sqrt{2})^2}{z+x} \geq \frac{(3(\sqrt{2}))^2}{2(x+y+z)} = \frac{9}{x+y+z}$$

which is the right hand side of the inequality.

5 Chebyshev inequality

5.1 Oriented sequences and Chebyshev inequality

This is another very useful and commonly used inequality which can be applied on oriented sequences (monotonically increasing or decreasing). There are two forms of this inequality, one for similarly oriented sequences, and one for oppositely oriented sequences. By similarly oriented we mean that the sequences are either both increasing or both decreasing. We will look at the case when they are both decreasing.

Let $a_1 \geq a_2 \geq \ldots \geq a_n$ and $b_1 \geq b_2 \geq \ldots \geq b_n$ be real numbers. Then we have:

$$n(a_1b_1 + a_2b_2 + \ldots + a_nb_n) \geq (a_1 + a_2 + \ldots + a_n)(b_1 + b_2 + \ldots + b_n)$$

Equality is achieved when either of the sequences is constant.

Let's see the case when the sequences are oppositely oriented.

Let $a_1 \geq a_2 \geq \ldots \geq a_n$ and $b_1 \leq b_2 \leq \ldots \leq b_n$ be real numbers. Then we have:

$$n(a_1b_1 + a_2b_2 + \ldots + a_nb_n) \leq (a_1 + a_2 + \ldots + a_n)(b_1 + b_2 + \ldots + b_n)$$

Equality is achieved when either of the sequences is constant.

As we can see, the only difference in the two theorems is the sign on the inequality and the orientation of the sequences. In many of the inequalities we can assume without loss of generality (WLOG) that the numbers are ordered. In other words, assuming that $a \geq b \geq c$ would give us the same result as if we assumed that $a \leq b \leq c$ or any other ordering. Let's see some examples.

5.2 Examples

Example 10 *Prove Nesbitt's inequality using Chebyshev inequality.*
Solution:
In case we forgot, let's remind ourselves of Nesbitt's inequality:

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq \frac{3}{2}$$

Let's denote the sum $a+b+c$ to be S. Then we can write the inequality as:

$$\frac{a}{S-a} + \frac{b}{S-b} + \frac{c}{S-c} \geq \frac{3}{2}$$

Note that we can assume without loss of generality that $a \geq b \geq c$. Then it is clear that $-a \leq -b \leq -c$ and if we add S to both sides we get $S-a \leq S-b \leq S-c$. Note that the sequences $a \geq b \geq c$ and $S-a \leq S-b \leq S-c$ are oppositely oriented. Now here comes the power of Chebyshev inequality for oppositely oriented sequences:

$$(\frac{a}{S-a} + \frac{b}{S-b} + \frac{c}{S-c})((S-a)+(S-b)+(S-c)) \geq 3(a+b+c)$$

Now, we have that $(S-a)+(S-b)+(S-c) = 2S = 2(a+b+c)$. The sum $(a+b+c)$ cancels with the one in right hand side and we get the desired inequality, namely:

$$\frac{a}{S-a} + \frac{b}{S-b} + \frac{c}{S-c} \geq \frac{3}{2}$$

or

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq \frac{3}{2}$$

which is what we wanted to prove.

Example 11 *Let a, b, c, d be positive real numbers. Prove the following:*

$$a^5 + b^5 + c^5 + d^5 \geq abcd(a+b+c+d)$$

Solution:
WLOG we can assume $a \geq b \geq c \geq d$. From this immediately follows that

$a^5 \geq b^5 \geq c^5 \geq d^5$. *The important thing is that these sequences are similarly oriented (we could have assumed that both of them are increasing instead). Then by Chebyshev inequality and AM-GM we get:*

$$4(a^5+b^5+c^5+d^5) \geq (a^4+b^4+c^4+d^4)(a+b+c+d) \geq 4\sqrt[4]{a^4b^4c^4d^4}(a+b+c+d) = 4abcd(a+b+c+d)$$

If we cancel the constant 4 on both sides we get the desired inequality, namely:

$$a^5 + b^5 + c^5 + d^5 \geq abcd(a + b + c + d)$$

Conclusion

We hope that this booklet was helpful to better understand the concepts covered on lectures and to make you see the beauty and the art of problem solving. However, we also hope that this is just a beginning of your journey with olympiad inequalities. There are many online resources that can help you continue developing your knowledge of this topic. Just google it! Good luck!